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LETTER TO THE EDITOR

Quantum deformations of SU(2)

D B Fairlie†

Institute for Advanced Study, Princeton, NJ 08540, USA

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Abstract. Explicit representations of two deformations of SU(2), one in the Cartan basis, and one in the Cartesian basis, are constructed. The Cartan deformation subsumes previously known cases, while the Cartesian deformation appears to be new. In the latter case, explicit reversible mappings in terms of representations of SU(2) and a Casimir for the algebra are constructed.

The first quantum deformations of the classical Lie algebras SU(N) [1], called quantised enveloping algebras by Drinfeld [2] and Jimbo [3], and referred to by the acronym QUE, treat the elements in the Cartan subalgebra on a different footing from the others. For SU(2), e.g., it is [1]

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = \frac{q^{2J_0} - q^{-2J_0}}{q^2 - q^{-2}}. \tag{1}$$

Recently some further examples of QUE algebras have been constructed by Woronowicz [4] and Witten [5] which treat all generators on a similar footing. Consequently these algebras may thus be a better start to discovering a quantum deformation of the Virasoro algebra [6]. In a subsequent article Curtright and Zachos [7] demonstrated explicit, almost always reversible, operator maps which are functionals of the SU(2) generators which transform any given representation of SU(2) into representations of the QUEs of (1) [8], Drinfeld, Jimbo, Woronowicz and Witten. They also apply their method to an obvious two-parameter extension which subsumes both Woronowicz's and Witten's forms. This algebra is

$$\begin{aligned} &rW_0 W_+ - 1/rW_+ W_0 = W_+ \\ &rW_- W_0 - 1/rW_0 W_- = W_- \\ &1/sW_+ W_- - sW_- W_+ = W_0. \end{aligned} \tag{2}$$

In the case of Witten's algebra, $r = \sqrt{s}$, while for Woronowicz $r = s^2$. The purpose of the present letter is to give an account of the representations of (2) from an elementary point of view, and to discuss a new quantum deformation of SU(2) in the Cartesian, rather than Cartan basis, i.e.

$$\begin{aligned} &qXY - 1/qYX = Z \\ &qYZ - 1/qZY = X \\ &qZX - 1/qXZ = Y. \end{aligned} \tag{3}$$

† On research leave from the University of Durham, UK.

This algebra is an egaliterian basis for the generators, and might be expected to possess attractive features in its representation theory. Unlike the classical case, where a change of basis is just a trivial linear transformation, there is no easy way to effect this in a QUE. A curious feature of the deformation maps for the algebra (3) is that they are most simply stated in terms of the Cartan, rather than Cartesian basis for SU(2). This observation is due to Zachos [9], and his construction of this map is presented below. There first follows a construction of the finite-dimensional reprentations of (2). They have been already given in terms of the representation matrices of SU(2) in [7], but are included here for the sake of completeness. Later we detail the representations of the Cartesian $SU(2)_q$, both as representation matrices and in terms of the representations of Cartan SU(2) and exhibit a Casimir, thus establishing (3) as a potential QUE. The remaining property is the existence of a co-multiplication, which permits the reduction of tensor products of representations. Since a deformation map from SU(2) exists, it is in principle possible to deduce a formula for combining representations of (3) by transformation of the co-multiplication for SU(2). This does not apparently result in a neat formula for the co-multiplication.

The way to find representations is to assume that W_0 is diagonal.

$$W_0=\begin{pmatrix} f_1 & 0 & \dots & 0 \\ 0 & f_2 & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & f_n \end{pmatrix}. \tag{4}$$

W_+, on the other hand is assumed to have non-zero entries only in the super diagonal;

$$W_+=\begin{pmatrix} 0 & \alpha_1 & 0 & \dots \\ 0 & 0 & \alpha_2 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots \end{pmatrix}. \tag{5}$$

We take for W_- the Hermitian conjugate of this expression. Then the first of the equations (2) determines f_i through the recurrence relation:

$$f_i=r^{-2}f_{i+1}+1/r \qquad i=1,2,\dots n.$$

The general solution is

$$f_i=f_n r^{2(i-n)}-\frac{(r^{2(i-n)}-1)r}{(r^2-1)}. \tag{6}$$

Notice that the simple linear combination is f_i-f_{i+1} given by

$$f_i-f_{i+1}=f_n r^{2(i-n)}(1-r^2)+r^{(2i-2n+1)}.$$

The unknown f_n is determined by the third equation of the set (2). We have

$$1/s\alpha_i^2-s\alpha_{i-1}^2=f_i \qquad \alpha_0=0. \tag{7}$$

There are fewer unknowns α than functions f_r, so there is a consistency check, namely

$$s^{2(n-1)}f_1+s^{2(n-2)}f_2+\dots+f_n=0. \tag{8}$$

This equation provides a normalisation for the value of f_n. The solution of these equations for f_n is

$$f_n=\frac{r}{r^2-1}\left(1-\frac{r^{2(n-1)}(r^2-s^2)(s^{2n}-1)}{(r^{2n}-s^{2n})(s^2-1)}\right). \tag{9}$$

Replacement of this expression in equation (7) gives

$$f_i = \frac{r}{r^2-1}\left(1 - \frac{r^{2(i-1)}(r^2-s^2)(s^{2n}-1)}{(r^{2n}-s^{2n})(s^2-1)}\right). \tag{10}$$

The sub/super diagonal entries in W_-/W_+ are given by

$$\mathrm{i}\sqrt{sf_1} \qquad \mathrm{i}\sqrt{s^3(f_1+s^{-2}f_2)} \qquad \mathrm{i}\sqrt{s^5(f_1+s^{-2}f_2+s^{-4}f_3)}\ldots \text{etc.} \tag{11}$$

Using (10) we can perform the partial summations and obtain a closed expression for α_i given by

$$\alpha_i = \left[\frac{rs}{(r^2-1)(s^2-1)}\left(\frac{(r^{2n}-1)(s^{2i}-1)-(s^{2n}-1)(r^{2i}-1)}{r^{2n}-s^{2n}}\right)\right]^{1/2}. \tag{12}$$

These expressions are equivalent to those in [7]. To make a comparison, n should be replaced by $2j+1$ and i by $j-j_0+1$.

The second case for which representations exist is $\mathrm{SU}(2)_q$ in the Cartesian basis, given by equations (3). The strategy in finding representations is to assume one matrix, Z say, is diagonal, with diagonal elements a_1, $a_2, \ldots a_n$, and also to assume that $a_i = -a_{n-i+1}$, just as in the representation theory of SU(2). X is taken to be with non-zero elements X_{ij} iff $i = j \pm 1$. Y may be thought of as determined by the third of equations (3). It is clearly not possible to have all three matrices anti-Hermitian unless q is a root of unity. Then the equations to determine a_j are

$$(q^2+q^{-2})a_i a_{i+1} = a_i^2 + a_{i+1}^2 + 1 \qquad 1 \leqslant i \leqslant n-1. \tag{13}$$

If n is even then the element $a_{n/2}$ is determined as $q\mathrm{i}/(q^2+1)$; if n is odd then $a_{(n+1)/2} = 0$, $a_{(n-1)/2} = \mathrm{i}$. The other elements are then obtained by the linear recursion;

$$(q^2+q^{-2})a_{i+1} = a_i + a_{i+2} \qquad \text{or better}$$

$$q_i - q^2 a_{i+1} = q^{-2}(a_{i+1} - q^2 a_{i+2}). \tag{14}$$

It is easy to solve this recursion relation; the solution which gives the appropriate central values is

$$a_i = -\mathrm{i}\left(\frac{q^{2i-n-1} - q^{-2i+n+1}}{q^2 - q^{-2}}\right) \qquad \text{for } 1 \leqslant i < n. \tag{15}$$

This solution is consistent with the quadratic relation (13). The matrix X takes the form

$$X = \begin{pmatrix} 0 & \beta_1 & 0 & \ldots & 0 \\ \beta_1 & 0 & \beta_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \beta_{n-2} & 0 & \beta_{n-1} \\ 0 & \ldots & 0 & \beta_{n-1} & 0 \end{pmatrix}. \tag{16}$$

The squares of the components are determined by the linear equations

$$(q^2+q^{-2})(a_{i-1}\beta_{i-1}^2 + a_{i+1}\beta_i^2) - 2(\beta_{i-1}^2 + \beta_i^2)a_i = a_i. \tag{17}$$

Equations (17) are subject to the restrictions $\beta_0 = \beta_n = 0$. There are additional equations which come from the off-diagonal components of the 'quommutator'. They are

$$(q^2+q^{-2})a_{i+1} - a_i - a_{i-1} = 0.$$

These equations are precisely the same as (14) above! They are thus automatically satisfied. Using the relations (14) we can rewrite the equations (17) in the simpler form

$$(a_{i-2}-a_i)\beta_{i-1}^2+(a_{i+2}-a_i)\beta_i^2=a_i \qquad i=1,n \tag{18}$$

i.e.

$$(q^{2i-n+1}+q^{-2i+n-1})\beta_i^2-(q^{2i-n-3}+q^{-2i+n+3})\beta_{i-1}^2=\frac{q^{2i-n-1}-q^{-2i+n+1}}{q^2-q^{-2}}. \tag{19}$$

This set of equations possesses the general solution

$$\beta_i^2=\frac{(q^{2(i-n)}-q^{-2(i-n)})(q^{2i}-q^{-2i})}{(q^{2i-n+1}+q^{-2i+n-1})(q^{2i-n-1}+q^{-2i+n+1})(q^2-q^{-2})^2} \qquad 1\leqslant i\leqslant n-1. \tag{20}$$

Setting $n=2j+1$, and taking the limit $q\to 1$, we recapture the familiar results for representations of SU(2). Of course these representation matrices are only determined up to a gauge transformation $U^{-1}XU$, $U^{-1}YU$, $U^{-1}ZU$.

In order to establish the representations as functionals of SU(2) generators as in [7] it is much better to express X, Y, Z in terms of SU(2) in a Cartan basis [9] than in the Cartesian one! Denoting the generators of SU(2) by $j_\pm$, j_0 where these are normalised to give

$$[j_0,j_+]=j_+ \qquad [j_-,j_0]=j_- \qquad [j_+,j_-]=j_0 \tag{21}$$

and denoting by j an operator given by the Casimir $j(j+1)=2j_+j_-+j_0(j_0+1)$, the operators Z, X, Y may be re-expressed as follows.

Equation (13) is translated as follows:

$$Z=\mathrm{i}\frac{q^{2j_0}-q^{-2j_0}}{q^2-q^{-2}} \tag{22}$$

$$X=\mathrm{i}v(j_0)j_++\mathrm{i}j_-v(j_0). \tag{23}$$

X is anti-Hermitian for real v, as it should be, and from the X, Z quommutator we get the presentation for

$$Y=-v(j_0)q^{2j_0-1}j_++j_-q^{1-2j_0}v(j_0). \tag{24}$$

Note this is also anti-Hermitian! The Z, Y quommutator now works automatically, whilst the X, Y one dictates

$$v(j_0)=\frac{-1}{(q^2-q^{-2})}\left[\frac{2(q^{2(j_0-j-1)}-q^{-2(j_0-j-1)})(q^{2(j_0+j)}-q^{-2(j_0+j)})}{(j+j_0)(j_0-j-1)(q^{2j_0}+q^{-2j_0})(q^{2j_0-2}+q^{-2j_0+2})}\right]^{1/2}. \tag{25}$$

These expressions are equivalent to (14) and (20) with the replacements $n=2j+1$ and the i, ith element of j_0, (when diagonalised) by $j-i+1$. This is the same transformation of variables as in the previous example.

This algebra possesses an appealing Casimir invariant which commutes with all the generators, and clearly reduces to the SU(2) case when $q=1$ and the quommutators reduce to commutators. It is [9]

$$(q^3+2/q)(XYZ+YZX+ZXY)-(q^{-3}+2q)(XZY+ZYX+YXZ). \tag{26}$$

The final requirement for a QUE is that there should exist a co-multiplication within the algebra which guarantees that the tensor product of representations reduces as a direct sum of representations. That something of this kind should exist is guaranteed by the invertible relations (22), (23) and (24). With their aid, the co-multiplication in SU(2) given by

$$\Delta(j_0)=j_0\otimes 1+1\otimes j_0 \qquad \Delta(j_\pm)=j_\pm\otimes 1+1\otimes j_\pm \tag{27}$$

may be expressed in terms of X, Y, Z; and since $\Delta(j_0)$, $\Delta(j_\pm)$ obey the same SU(2) commutation relations as j_0, $j_\pm$, then so will $\Delta(Z)=(q^{2\Delta(j_0)}-q^{-2\Delta(j_0)})/(q^2-q^{-2})$ etc obey the quommutation relations (3). This procedure gives a relatively simple construction for $\Delta(Z)$:

$$\Delta(Z)=Z\otimes Z'+Z'\otimes Z$$

where

$$Z'=\frac{q^{2j_0}+q^{-2j_0}}{2}=\tfrac{1}{2}\sqrt{4-Z^2(q^2-q^{-2})^2}. \tag{28}$$

It is easy to verify that in a diagonal basis, $\Delta(Z)$ decomposes into the direct sum of representations. It seems that neither $\Delta(X)$ nor $\Delta(Y)$ can be expressed in a similar simple fashion.

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